

# Fibred coarse embeddability of box spaces and proper isometric affine actions on $L^p$ spaces

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## Abstract

We show the necessary part of the following theorem : a finitely generated, residually finite group has property  $PL^p$  (i.e. it admits a proper isometric affine action on some  $L^p$  space) if, and only if, one (or equivalently, all) of its box spaces admits a fibred coarse embedding into some  $L^p$  space. We also prove that coarse embeddability of a box space of a group into a  $L^p$  space implies property  $PL^p$  for this group.

## 1 Introduction

The notion of fibred coarse embeddings into Hilbert space, which generalizes the notion of coarse embeddings, has been introduced by Chen, Wang and Yu in [CWY13] to provide a tool for the study of the maximal Baum-Connes conjecture. They proved in this paper that any metric space with bounded geometry admitting a fibred coarse embedding into a Hilbert space satisfies the maximal coarse Baum-Connes conjecture. In [CWW13], Chen, Wang and Wang characterized the Haagerup property in terms of fibred coarse embedding into Hilbert space : in fact, they showed that a finitely generated, residually finite group has the Haagerup property if, and only if, one of its box space admits a fibred coarse embedding into a Hilbert space. The goal of this note is to extend this result to the class of  $L^p$  spaces (for a fixed  $p \geq 1$ ).

**Theorem 1.1.** *Let  $\Gamma$  be a finitely generated, residually finite group,  $(\Gamma_i)_{i \in N^*}$  be a nested sequence of finite index normal subgroups of  $\Gamma$  with trivial intersection and  $1 \leq p \leq \infty$ . Then  $\Gamma$  has property  $PL^p$  if, and only if, the box space  $\square_{\{\Gamma_i\}}\Gamma$  admits a fibred coarse embedding into a  $L^p$  space.*

We also prove the following proposition which extends to  $L^p$  spaces a result of Roe in the setting of Hilbert spaces (see [Roe03]).

**Proposition 1.2.** *Let  $\Gamma$  be a finitely generated, residually finite group,  $(\Gamma_i)_{i \in N^*}$  be a nested sequence of finite index normal subgroups of  $\Gamma$  with trivial intersection and  $1 \leq p \leq \infty$ . If the box space  $\square_{\{\Gamma_i\}}\Gamma$  admits a coarse embedding into a  $L^p$  space, then  $\Gamma$  has property  $PL^p$ .*

Theorem 1.1 and Proposition 1.2 can be stated for other classes of Banach spaces instead of  $L^p$  spaces. In fact, the proof of the necessary condition (see Proposition 3.4) and the proof of Proposition 1.2 only uses the fact that the class of  $L^p$  spaces (for a fixed  $1 \leq p < \infty$ ) is a class  $\mathcal{B}$  of Banach spaces satisfying the following properties :

1.  $\mathcal{B}$  is closed under taking some particular normed finite powers i.e. :

for every  $n \in \mathbb{N}^*$  and every  $B \in \mathcal{B}$ , there exists a norm  $N$  on  $\mathbb{R}^n$  such that :

- there exists  $c \geq 0$  such that, for all  $K, K' \geq 0$  the  $n$ -cube  $\{x \in \mathbb{R}^n \mid K \leq x_i \leq K'\}$  is contained in the annulus  $\{x \in \mathbb{R}^n \mid cK \leq N(x) \leq cK'\}$  - or, in other words, for all  $x \in \mathbb{R}^n$ , if the components of  $x$  are controlled below by  $K$  and above by  $K'$  then so does  $\frac{1}{c}N(x)$ ;

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- the Banach space  $B^n$  endowed with the norm  $\|\cdot\| = N(\|\pi_1(\cdot)\|_B, \dots, \|\pi_n(\cdot)\|_B)$  belongs to  $\mathcal{B}$  (where  $\pi_i$  is the canonical projection of  $B^n$  on its  $i$ -th factor).

In the  $L^p$  case, for  $n \in \mathbb{N}^*$ , the norm of  $\ell_p^n = \ell^p(\{1, \dots, n\})$  fits, and  $c = n^{\frac{1}{p}}$ .

2.  $\mathcal{B}$  is closed under ultraproducts (see Definition 3.2).

In the  $L^p$  case, the stability by ultraproduct is a result due to Krivine (see [Kri67] Theorem 1 and its application p.17).

For a class of Banach spaces  $\mathcal{B}$ , property  $P\mathcal{B}$  is an analog of the Haagerup property viewed with the Gromov's definition of a-T-menability (definition in terms of isometric affine actions, see [Gro93] or [CCJ<sup>+</sup>01]) where the class of Hilbert spaces is replaced by the class  $\mathcal{B}$ . One of the motivation in the study of this property is given by a result of Kasparov and Yu in [KY12] which asserts that groups admitting coarse embeddings into uniformly convex Banach spaces satisfy the Novikov conjecture (in particular, groups having property  $P\mathcal{B}$  where  $\mathcal{B}$  is a subclass of uniformly convex Banach spaces admit such embeddings).

An *isometric affine action* of a group  $\Gamma$  on a Banach space  $B$  is a morphism  $\alpha$  of  $\Gamma$  into the group  $\text{Aff}(B) \cap \text{Isom}(B)$  of affine isometric transformations of  $B$ ; such an action can be characterized by the following decomposition :

$$\alpha(g)v = \pi(g)v + b(g), \text{ for all } g \in \Gamma, v \in B,$$

where  $\pi$  is an isometric representation of  $\Gamma$  on  $B$  and  $b$  is a 1-cocycle with respect to  $\pi$  i.e., for all  $g, h \in \Gamma$ ,  $b(gh) = \pi(g)b(h) + b(g)$ .

The action  $\alpha$  is said to be *proper* if  $\|b(g)\|_B \xrightarrow[g \rightarrow \infty]{} +\infty$ .

**Definition 1.3.** Let  $\mathcal{B}$  be a class of Banach spaces. A (discrete) group  $\Gamma$  is said to have property  $P\mathcal{B}$  if there exists a proper isometric affine action of  $\Gamma$  on some Banach space  $B \in \mathcal{B}$ .

Many recent progress has been made in the study of isometric affine actions on Banach spaces, and more particularly in the case of  $L^p$  spaces for a fixed  $1 \leq p \leq \infty$ . Bader, Furman, Gelander, Monod studied the relationships between two different generalizations of Kazhdan's property ( $T$ ), namely property  $FL^p$  and property  $(T_{L^p})$  in [BFGM07]. On the other hand, property  $PL^p$ , also referred as a- $FL^p$ -menability by some authors, is a strong negation of property  $FL^p$ . Examples of  $PL^p$  groups are given by [Yu05], where Yu proved that, for a discrete hyperbolic group  $\Gamma$ , there exists  $2 \leq p_0 < \infty$  such that  $\Gamma$  has property  $PL^p$  for all  $p \geq p_0$ ; or by [CTV08], where Cornulier, Tessera, Valette showed that the hyperbolic simple Lie group  $\text{Sp}(n, 1)$  has property  $PL^p$  for all  $p > 4n + 2$ . We give here an overview of what is known about the links between property  $PL^p$  and  $PL^q$  for various values of  $p$  and  $q$  :

- |     |                                |                   |                                    |
|-----|--------------------------------|-------------------|------------------------------------|
| (1) | Haagerup ( $=PL^2$ )           | $\Rightarrow$     | $PL^p$ for all $0 < p \leq \infty$ |
| (2) | $PL^p$ for some $0 < p \leq 2$ | $\Leftrightarrow$ | Haagerup                           |
| (3) | $PL^p$ for some $p > 2$        | $\Rightarrow$     | Haagerup                           |
| (4) | $PL^p$ for some $p > 2$        | $\Rightarrow$     | $PL^q$ for all $q > p$             |

Implication (1) was proved in [CMV04] by Cherix, Martin and Valette for countable discrete groups, using the notion of spaces with measured walls. Equivalence (2) follows from results of Delorme-Guichardet ([Gui72], [Del77]) and Akemann-Walter ([AW81]). See [CDH10] Corollary 1.5 and Remark 1.6 for proofs and discussions about (1) and (2) in the setting of second countable, locally compact groups.

Assertion (3) follows from the fact that a discrete hyperbolic group with property ( $T$ ) fails the Haagerup property but has  $PL^p$  for some  $p > 2$  by the result of Yu quoted before. We mention that assertion (4) is still an open question which appears in [CDH10], Question 1.8.

Concerning stability, property  $PL^p$  (for a fixed  $p > 2$ ) is closed under taking closed subgroups, direct sums, amalgamated free products over finite subgroups (see [Pil15] and [Arn13] for proofs of this result with different approaches) but it is not stable by extension in general. However, using a construction of Cornulier, Stalder and Valette in [CSV12], the author showed in [Arn14] that property  $PL^p$  is closed

under wreath product by Haagerup groups. We would like to mention that Haagerup property is stable by amenable extensions, but for property  $PL^p$  with  $p > 2$ , it remains an open problem.

**Remark 1.4.** Notice that unlike in the Hilbert spaces case, property  $PL^p$  is no longer equivalent to property  $HL^p$  i.e. the existence of a  $C_0$  representation on some  $L^p$  space which almost has invariant vectors. For instance, a discrete hyperbolic group with property (T) has property  $PL^p$  for some  $p > 2$ , but it also has property  $(T_{L^p})$  for all  $p \geq 1$  (see [BFGM07]) which is a strong negation of property  $HL^p$ .

**Definition 1.5.** Let  $\Gamma$  be a finitely generated, residually finite group and let  $\Gamma_1 \triangleright \dots \triangleright \Gamma_i \triangleright \dots$  be a nested sequence of finite index normal subgroups of  $\Gamma$  such that  $\bigcap_{i=1}^{\infty} \Gamma_i = \{e\}$ . The box space associated with the sequence  $\{\Gamma_i\}_{i \in \mathbb{N}^*}$ , denoted by  $\square_{\{\Gamma_i\}} \Gamma$  or simply  $\square \Gamma$ , is the coarse disjoint union  $\bigsqcup_{i=1}^{\infty} \Gamma / \Gamma_i$  of the finite quotient groups, i.e., the disjoint union where each quotient is endowed with the metric induced by the image of the generating set of  $\Gamma$ , and the distances between the identity elements of two successive quotients are chosen to be greater than the maximum of their diameters.

There is a large spectrum of analytic properties of a group  $\Gamma$  which link to geometric properties of its box space  $\square \Gamma$ . As in [CWW13], we summarize here different correspondances :

$$\begin{aligned} \Gamma \text{ amenable} &\Leftrightarrow \square \Gamma \text{ Property A} \\ \Gamma \text{ Property (T)} &\Leftrightarrow \square \Gamma \text{ geometric Property (T)} \\ \Gamma \text{ Haagerup} &\Leftrightarrow \square \Gamma \text{ fibred coarsely embeddable into Hilbert space} \\ \Gamma \text{ Property } PL^p &\Leftrightarrow \square \Gamma \text{ fibred coarsely embeddable into some } L^p \\ \Gamma \text{ Property } PL^p &\Leftarrow \square \Gamma \text{ coarsely embeddable into some } L^p \end{aligned}$$

The first equivalence was established by Roe in [Roe03] where Property A is a non-equivariant version of amenability defined by Yu ([Yu00]) which guarantees coarse embeddability into Hilbert spaces. The second one is due to Willett and Yu in [WY12] where they introduced the notion of geometric property (T). For a coarse disjoint union of finite graphs, geometric property (T) implies the property of being an expander. The third equivalence is the result of Chen, Wang and Wang ([CWW13]) mentioned in the introduction.

The last two assertions are proved in the present note. In [Roe03], Roe established the last implication in the Hilbert case ( $p = 2$  case); and notice that the converse implication fails. In fact, on one hand, the free group on two generators has the Haagerup property, and on the other hand, it has property  $(\tau)$  with respect to some sequences of finite index normal subgroups (see [Lub10]): hence, the associated box spaces are expanders, which implies that they are not coarsely embeddable into Hilbert space.

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## 2 Fibred Coarse embeddings into Banach spaces

We recall here the notion of coarse embedding and the notion of fibred coarse embedding introduced in [CWY13] where the notion of Banach spaces replaces the original Hilbert spaces model.

**Definition 2.1.** Let  $(X, d)$  be a metric space and  $B$  be a Banach space. A map  $f : X \rightarrow B$  is said to be a coarse embedding of  $X$  into  $B$  if there exist two non-decreasing functions  $\rho_1$  and  $\rho_2$  from  $[0, +\infty)$  to  $(-\infty, +\infty)$  with  $\lim_{r \rightarrow +\infty} \rho_i(r) = +\infty$  for  $i = 1, 2$ , such that, for all  $x, y \in X$  :

$$\rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y)).$$

**Remark 2.2.** Every metric space  $(X, d)$  admits a coarse embedding into  $\ell^\infty(X)$  via, for a fixed  $x_0 \in X$ , the map

$$f : x \rightarrow \{y \mapsto d(x, y) - d(x_0, y)\}.$$

In fact,  $f$  is an isometric embedding.

Moreover, for a finitely generated group  $\Gamma$  endowed with the word metric  $d$  induced by a finite generating set, the same map  $f : g \mapsto \{h \mapsto d(g, h) - d(e_\Gamma, h)\}$  is a proper cocycle with respect to the left regular representation on  $\ell^\infty(\Gamma)$ . Hence, every finitely generated group has property  $PL^\infty$ .

**Definition 2.3.** A metric space  $(X, d)$  is said to admit a fibred coarse embedding into a Banach space  $B$ , if there exist :

1. a field of Banach spaces  $(B_x)_{x \in X}$  over  $X$  such that each  $B_x$  is affinely isometric to  $B$  ;
2. a section  $s : X \rightarrow \bigsqcup_{x \in X} B_x$  (i.e.  $s(x) \in B_x$ ) ;
3. two non-decreasing functions  $\rho_1$  and  $\rho_2$  from  $[0, +\infty)$  to  $(-\infty, +\infty)$  with  $\lim_{r \rightarrow +\infty} \rho_i(r) = +\infty$  for  $i = 1, 2$  such that :  
for any  $r > 0$ , there exists a bounded subset  $K_r \subset X$  for which there exists a “trivialization”

$$t_C : (B_x)_{x \in C} \rightarrow C \times B$$

for each subset  $C \subset X \setminus K_r$  of diameter less than  $r$  ; that is, a map from  $(B_x)_{x \in C}$  to the constant field  $C \times B$  over  $B$  such that the restriction to the fibre  $B_x$  for  $x \in C$  is an affine isometry  $t_C : B_x \rightarrow B$ , satisfying the following conditions :

- i) for any  $x, y \in C$ ,  $\rho_1(d(x, y)) \leq \|t_C(x)(s(x)) - t_C(y)(s(y))\|_B \leq \rho_2(d(x, y))$  ;
- ii) for any two subsets  $C_1, C_2 \subset X \setminus K_r$  of diameter less than  $r$  with  $C_1 \cap C_2 \neq \emptyset$ , there exists an affine isometry  $t_{C_1 C_2} : B \rightarrow B$  such that  $t_{C_1}(x) \circ t_{C_2}(x)^{-1} = t_{C_1 C_2}$ , for all  $x \in C_1 \cap C_2$ .

**Remark 2.4.** Let  $(X, d)$  be a metric space and  $B$  be a Banach space. If  $X$  coarsely embeds into  $B$  then  $X$  fibred coarsely embeds into  $B$ . In fact, if  $f : X \rightarrow B$  is a coarse embedding with control functions  $\rho_1, \rho_2$  then a fibred coarse embedding of  $X$  into  $B$  is given by :

1. the field of Banach spaces  $(B_x)_{x \in X}$  where  $B_x := B$  for all  $x \in X$  ;
2. the section  $s : x \mapsto f(x) \in B = B_x$  ;
3. the two control functions  $\rho_1$  and  $\rho_2$  and for each  $r > 0$ , considering  $K_r = \emptyset$ , for all  $C$  of diameter less than  $r$ , the “trivial” trivialisation given by, for  $x \in X$ ,  $t_C(x) = Id_B$  (which satisfies condition i) and ii) since  $f$  is a coarse embedding).

The following proposition is proved by Chen, Wang and Wang in [CWW13] (see Proposition 1.4) in the general setting of fibred coarse embeddings into metric spaces.

**Proposition 2.5.** Let  $\Gamma$  be a finitely generated, residually finite group. If  $\Gamma$  acts properly isometrically on a metric space  $Y$ , then any box space  $\square \Gamma$  admits a fibred coarse embedding into  $Y$ .

We can then reformulate this statement in the context of property  $P\mathcal{B}$  :

**Corollary 2.6.** Let  $\Gamma$  be a finitely generated, residually finite group and  $\mathcal{B}$  a class of Banach spaces. If  $\Gamma$  has property  $P\mathcal{B}$ , then any box space  $\square \Gamma$  admits a fibred coarse embedding into some Banach space  $B \in \mathcal{B}$ .

### 3 Proof of the main results

**Definition 3.1.** Let  $\Gamma$  be a finitely generated group and  $r$  be a non-negative real.

i) Let  $X$  be a set. A map  $\alpha : \Gamma \times X \rightarrow X$  is said to be a  $r$ -locally action of  $G$  on  $X$  if :

- for all  $g \in \Gamma$  such that  $d(e, g) < r$ ,  $\alpha(g) : X \rightarrow X$  is a bijection;
- for all  $g, h \in \Gamma$  such that  $d(e, g), d(e, h), d(e, gh)$  are less than  $r$ ,

$$\alpha(gh) = \alpha(g)\alpha(h).$$

ii) Let  $B$  be a Banach space. A map  $\pi : \Gamma \times B \rightarrow B$  is said to be a  $r$ -locally isometric representation of  $G$  on  $B$  if  $\pi$  is a  $r$ -locally action of  $\Gamma$  on  $B$  and for all  $g \in \Gamma$  such that  $d(e, g) < r$ ,  $\pi(g) : B \rightarrow B$  is a linear isometry.

In this case, a map  $b : \Gamma \rightarrow B$  such that, for all  $g, h \in \Gamma$  such that  $d(e, g), d(e, h), d(e, gh)$  are less than  $r$ ,  $\pi(g)b(h) + b(g) = b(gh)$ , is called a  $r$ -locally cocycle with respect to  $\pi$ .

iii) Let  $B$  be a Banach space. A map  $\alpha : G \times B \rightarrow B$  is called a  $r$ -locally isometric affine action of  $\Gamma$  on  $B$  if it can be written as  $\alpha(g)\cdot = \pi(g)\cdot + b(g)$  where  $\pi$  is a  $r$ -locally isometric representation and  $b$  is a  $r$ -locally cocycle with respect to  $\pi$ .

Using the notion of ultrafilters and ultraproducts, one can build a global isometric affine action from a family of  $r$ -locally isometric affine actions with  $r \rightarrow +\infty$ .

Let  $\mathcal{U}$  be a non-principal ultrafilter on  $\mathbb{N}^*$  i.e.  $\mathcal{U}$  is a subset of  $\mathcal{P}(\mathbb{N}^*)$  stable by intersection such that :

- the empty set  $\emptyset$  does not belong to  $\mathcal{U}$ ,
- for all  $A, B \in \mathcal{P}(X)$  such that  $A \subset B$ ,  $A \in \mathcal{U}$  implies  $B \in \mathcal{U}$ ,
- for all  $A \in \mathcal{P}(X)$ ,  $A \in \mathcal{U}$  or  $X \setminus A \in \mathcal{U}$ .
- finite subsets of  $\mathbb{N}^*$  do not belong to  $\mathcal{U}$ .

The  $\mathcal{U}$ -limit of bounded real valued sequence  $(x_r)_{r \in \mathbb{N}^*}$  is the unique  $x \in \mathbb{R}$  denoted by  $\lim_{\mathcal{U}} x_r$  such that for all  $\varepsilon > 0$ , the set  $\{r \in \mathbb{N}^* \mid |x_r - x| \leq \varepsilon\}$  belongs to  $\mathcal{U}$ .

**Definition 3.2.** Let  $(B_r)_{r \in \mathbb{N}^*}$  be a family of Banach spaces and consider the space  $\ell^\infty(\mathbb{N}^*, (B_r)_{r \in \mathbb{N}^*})$  of sequences  $(a_r)_{r \in \mathbb{N}^*}$  satisfying that there exists  $K \geq 0$  such that for all  $r \in \mathbb{N}^*$ ,  $a_r \in B_r$  with  $\|a_r\|_{B_r} \leq K$ .

The ultraproduct  $B_{\mathcal{U}}$  of the family  $(B_r)_{r \in \mathbb{N}^*}$  with respect to a non-principal ultrafilter  $\mathcal{U}$  is the closure of the space  $\ell^\infty(\mathbb{N}^*, (B_r)) / \sim_{\mathcal{U}}$  endowed with the norm  $\|(a_r)\|_{B_{\mathcal{U}}} := \lim_{\mathcal{U}} \|a_r\|_{B_r}$  where, for  $(a_r), (b_r) \in \ell^\infty(\mathbb{N}^*, (B_r))$ ,

$$(a_r) \sim_{\mathcal{U}} (b_r) \text{ if, and only if, } \|(a_r) - (b_r)\|_{B_{\mathcal{U}}} = 0.$$

**Lemma 3.3.** Let  $\Gamma$  be a finitely generated group,  $(B_r)_{r \in \mathbb{N}^*}$  be a family of Banach spaces and  $B_{\mathcal{U}}$  be the ultraproduct of the family  $(B_r)$  with respect to a non-principal ultrafilter  $\mathcal{U}$  on  $\mathbb{N}^*$ . For each  $r \in \mathbb{N}^*$ , assume that  $\Gamma$  admits a  $r$ -locally isometric affine action  $\alpha_r$  on  $B_r$  with  $\alpha_r(g)\cdot = \pi_r(g)\cdot + b_r(g)$ .

If, for all  $g \in \Gamma$ ,  $(b_r(g))_{r \in \mathbb{N}^*}$  belongs to  $B_{\mathcal{U}}$ , then there exists an isometric affine action  $\alpha$  of  $G$  on  $B_{\mathcal{U}}$  of the family  $(B_r)$  such that  $\alpha(g)\cdot = \pi(g)\cdot + b(g)$  where  $\pi$  is an isometric representation of  $\Gamma$  on  $B_{\mathcal{U}}$  and  $b : G \rightarrow B_{\mathcal{U}}$  is a cocycle with respect to  $\pi$  satisfying, for  $g \in \Gamma$  :

$$b(g) = (b_r(g))_{r \in \mathbb{N}^*}.$$

*Proof.* For  $g \in \Gamma$ , we define  $\pi(g) : B_{\mathcal{U}} \rightarrow B_{\mathcal{U}}$  by, for  $a = (a_r)_{r \in \mathbb{N}^*} \in B_{\mathcal{U}}$ ,

$$\pi(g)a = (\pi_r(g)a_r)_{r \in \mathbb{N}^*};$$

and we set  $b(g) = (b_r(g))_{r \in \mathbb{N}^*} \in B_{\mathcal{U}}$ .

Let  $g, h \in \Gamma$ . For all  $r \in \mathbb{N}^*$  such that  $r > \max(d(e, g), d(e, h), d(e, gh))$ , we have, for all  $(a_r) \in B_{\mathcal{U}}$ ,  $\pi_r(g)\pi_r(h)a_r = \pi_r(gh)a_r$  and then the set  $\{r \in \mathbb{N}^* \mid \pi_r(g)\pi_r(h)a_r = \pi_r(gh)a_r\}$  belongs to  $\mathcal{U}$ . Hence,

for all  $g, h \in \Gamma$ ,  $\pi(g)\pi(h) = \pi(gh)$ . Now, for  $g \in \Gamma$ , since for all  $r$  large enough,  $\pi_r(g)$  is an isometric isomorphism of  $B_r$ , it follows, by a similar argument, that  $\pi(g)$  is an isometric isomorphism of  $\mathcal{B}_{\mathcal{U}}$ .

Thus,  $\pi$  is an isometric representation of  $\Gamma$  on  $B_{\mathcal{U}}$ .

Let  $g, h \in \Gamma$ . For all  $r \in \mathbb{N}^*$  such that  $r > \max(d(e, g), d(e, h), d(e, gh))$ , we have  $b_r(gh) = \pi_r(g)b_r(h) + b_r(g)$ . Hence, for all  $g, h \in \Gamma$ ,  $b(gh) = \pi(g)b(h) + b(g)$  and then,  $b$  is a cocycle with respect to  $\pi$ . It follows that the map  $\alpha$  such that  $\alpha(g)\cdot = \pi(g)\cdot + b(g)$  is an isometric affine action of  $\Gamma$  on  $B_{\mathcal{U}}$ .  $\square$

*Proof of Proposition 1.2.* The case  $p = \infty$  is trivial (see Remark 2.2).

Let  $1 \leq p < \infty$ . Let  $\{\Gamma_n\}_{n \in \mathbb{N}^*}$  be a nested sequence of finite index normal subgroups of  $\Gamma$  with trivial intersection such that the associated box space  $\square\Gamma$  admits a coarse embedding  $f$  into a  $L^p$  space denoted by  $B$  with control functions  $\rho_1, \rho_2$ .

Let  $n \in \mathbb{N}^*$  and denote  $X_n := \Gamma/\Gamma_n$ . Let us consider the Banach space  $\bigoplus_{z \in X_n} B$  endowed with the following norm : for a vector  $\xi = \bigoplus_{z \in X_n} \xi_z$ ,

$$\|\xi\|_p = \left( \sum_{z \in X_n} \|\xi_z\|_B^p \right)^{\frac{1}{p}}.$$

For  $x \in X_n$ , we define the following vector of  $\bigoplus_{z \in X_n} B$  :

$$\tilde{b}_n(x) := \frac{1}{(\#X_n)^{\frac{1}{p}}} \bigoplus_{z \in X_n} (f(zx) - f(z));$$

and let  $\tilde{\sigma}_n$  be the isometric representation of  $X_n$  on  $\bigoplus_{z \in X_n} B$  such that for  $\xi = \bigoplus_{z \in X_n} \xi_z$ ,

$$\tilde{\sigma}_n(x)\xi = \bigoplus_{z \in X_n} \xi_{zx}.$$

Then  $\tilde{b}_n : X_n \rightarrow \bigoplus_{z \in X_n} B$  is a cocycle with respect to  $\tilde{\sigma}_n$ . In fact, since for  $x, y, z \in X_n$ , we have  $f(zxy) - f(z) = (f(zxy) - f(zx)) + (f(zx) - f(z))$ , it follows that :

$$\begin{aligned} \tilde{b}_n(xy) &= \frac{1}{(\#X_n)^{\frac{1}{p}}} \bigoplus_{z \in X_n} (f(zxy) - f(zx)) + \frac{1}{(\#X_n)^{\frac{1}{p}}} \bigoplus_{z \in X_n} (f(zx) - f(z)), \\ \tilde{b}_n(xy) &= \tilde{\sigma}_n(x)\tilde{b}_n(y) + \tilde{b}_n(x). \end{aligned}$$

Moreover, since  $f$  is a coarse embedding, we have, for all  $x \in X_n$  :

$$\rho_1(d_{X_n}(x, e)) \leq \|\tilde{b}_n(x)\|_p \leq \rho_2(d_{X_n}(x, e)),$$

where  $e$  is the identity element of  $X_n$ .

Now, for each  $r \in \mathbb{N}^*$ , choose  $n_r$  such that the canonical quotient map  $\pi_{n_r} : \Gamma \twoheadrightarrow X_{n_r}$  is  $r$ -isometric and define  $\sigma_r := \tilde{\sigma}_{n_r} \circ \pi_{n_r}$  and  $b_r := \tilde{b}_{n_r} \circ \pi_{n_r}$ . Thus, for every  $r$ ,  $b_r$  is a cocycle with respect to the isometric representation  $\sigma_r$  of  $\Gamma$  on  $\bigoplus_{z \in X_{n_r}} B$  and we have, for  $g \in \Gamma$  such that  $d_{\Gamma}(g, e_{\Gamma}) < r$  :

$$\rho_1(d_{\Gamma}(g, e)) \leq \|b_r(g)\|_p \leq \rho_2(d_{\Gamma}(g, e_{\Gamma})). \quad (*)$$

Let  $\mathcal{U}$  be a non-principal ultrafilter on  $\mathbb{N}^*$  and  $B_{\mathcal{U}}$  be the ultraproduct of  $\left( \bigoplus_{z \in X_{n_r}} B \right)_{r \in \mathbb{N}^*}$ . For each  $r$ , the map  $\alpha_r$  defined by  $\alpha_r(g)\cdot := \pi_{n_r}(g)\cdot + b_r(g)$  for  $g \in \Gamma$ , is an isometric affine action. By  $(*)$ , for all  $g \in \Gamma$ ,  $(b_r(g))_{r \in \mathbb{N}^*}$  belongs to  $B_{\mathcal{U}}$ .

Hence, by Lemma 3.3, there exists an isometric affine action  $\alpha$  of  $\Gamma$  on  $B_{\mathcal{U}}$  such that  $b : g \mapsto (b_r(g))$  is a cocycle for this action. Moreover, for  $g \in \Gamma$ , since for all  $r$  large enough,  $\rho_1(d_{\Gamma}(g, e)) \leq \|b_r(g)\|_p$ , we have :

$$\rho_1(d_{\Gamma}(g, e)) \leq \|b(g)\|_{B_{\mathcal{U}}};$$

hence  $\alpha$  is proper. As the class of  $L^p$  spaces is closed under  $p$ -normed powers and ultraproduct, it follows that  $\Gamma$  has property  $PL^p$ .  $\square$

For the next proposition, the steps of the proof are essentially the same as in the proof of Proposition 1.2. But, in this case, since for a given constant  $r$ , the trivialization of a fibred coarse embedding is defined on subsets of diameter less than  $r$ , we need to "r-localize" our construction of isometric affine actions of the quotient groups  $\Gamma/\Gamma_{nr}$ .

**Proposition 3.4.** *Let  $1 \leq p \leq \infty$  and let  $\Gamma$  be a finitely generated, residually finite group. If a box space  $\square\Gamma$  of  $\Gamma$  admits a fibred coarse embedding into some  $L^p$  space, then  $\Gamma$  has property  $PL^p$ .*

*Proof.* The case  $p = \infty$  is trivial (see Remark 2.2).

Let  $1 \leq p < \infty$ . Let  $\{\Gamma_n\}_{n \in \mathbb{N}^*}$  be a nested sequence of finite index normal subgroups of  $\Gamma$  with trivial intersection such that the associated box space  $\square\Gamma$  admits a fibred coarse embedding into a  $L^p$  space denoted by  $B$ .

We set  $X_n = \Gamma/\Gamma_n$  and  $X = \bigsqcup_{n \in \mathbb{N}^*} X_n (= \square\Gamma)$ . Let  $r \in \mathbb{N}^*$ . By Definition 2.3, there exist  $K_r$  and a trivialization  $t_C$  for each  $C \subset X \setminus K_r$  of diameter less than  $r$  satisfying conditions i) and ii).

Now, choose  $n_r$  large enough such that  $X_{n_r} \subset X \setminus K_r$  and the quotient map  $\pi_{n_r} : \Gamma \rightarrow X_{n_r}$  is  $r$ -isometric, i.e., for each subset  $Y \subset \Gamma$  of diameter less than  $r$ ,  $(\pi_{n_r})|_Y$  is an isometry onto its image.

For  $z \in X_{n_r}$ , we denote by  $C_z := \{x \in X_{n_r} \mid d_{X_{n_r}}(z, x) < r\}$  the  $r$ -ball centered in  $z$  of  $X_{n_r}$  and we set, for  $x \in X_{n_r}$ , the following vector  $c_r^z(x)$  of  $B$  :

$$c_r^z(x) := \begin{cases} t_{C_z}(z)(s(z)) - t_{C_z}(zx)(s(zx)) & \text{if } d_{X_{n_r}}(e, x) < r \text{ (i.e } x \in C_e\text{);} \\ 0 & \text{otherwise,} \end{cases}$$

where  $e$  is the identity element of  $X_{n_r}$ . Notice that, by Definition 2.3 3. i) for any  $z \in X_{n_r}$  and any  $x \in C_e$ ,  $\rho_1(d_{X_{n_r}}(e, x)) \leq \|c_r^z(x)\|_B \leq \rho_2(d_{X_{n_r}}(e, x))$ .

Let us consider the map  $\tilde{b}_r : X_{n_r} \rightarrow \bigoplus_{z \in X_{n_r}} B$ , defined by, for  $x \in X_{n_r}$  :

$$\tilde{b}_r(x) = \frac{1}{(\#X_{n_r})^{\frac{1}{p}}} \bigoplus_{z \in X_{n_r}} c_r^z(x).$$

We endow  $\bigoplus_{z \in X_{n_r}} B$  with the norm induced by the norm of  $\ell_p^n$  i.e. for  $\xi = \bigoplus_{z \in X_{n_r}} \xi_z$ ,

$$\|\xi\|_p = \left( \sum_{z \in X_{n_r}} \|\xi_z\|_B^p \right)^{\frac{1}{p}}.$$

Hence, for  $x \notin C_e$ ,  $\tilde{b}_r(x)$  vanishes, and for  $x \in C_e$ ,  $\rho_1(d_{X_{n_r}}(e, x)) \leq \|\tilde{b}_r(x)\|_p \leq \rho_2(d_{X_{n_r}}(e, x))$ .

We claim that  $\tilde{b}_r(x)$  is a  $r$ -locally cocycle for a  $r$ -locally isometric representation  $\tilde{\sigma}_r$  that we define as follows :

For  $x \in C_e$  and  $z \in X_{n_r}$ , let  $\rho_{C_z C_{zx}}$  be the linear part of the affine isometry  $t_{C_z C_{zx}} : B \rightarrow B$ . We define  $\tilde{\sigma}_r(x) : \bigoplus_{z \in X_{n_r}} B \rightarrow \bigoplus_{z \in X_{n_r}} B$  by, for  $\xi = \bigoplus_{z \in X_{n_r}} \xi_z$  :

$$\tilde{\sigma}_r(x)(\xi) := \begin{cases} \bigoplus_{z \in X_{n_r}} \rho_{C_z C_{zx}}(\xi_{zx}) & \text{if } x \in C_e; \\ \xi & \text{otherwise.} \end{cases}$$

The map  $\tilde{\sigma}_r$  is indeed a  $r$ -locally isometric representation : it is clear that  $\tilde{\sigma}_r(x)$  is a isometric isomorphism for all  $x \in X_{n_r}$ ; moreover it follows from Definition 2.3 3. ii) that  $t_{C_z C_{zy}} \circ t_{C_{zy} C_{zyx}} = t_{C_z C_{zyx}}$  for all  $x, y \in C_e$  with  $d_{X_{n_r}}(e, yx) < r$ , and then,  $\rho_{C_z C_{zy}} \circ \rho_{C_{zy} C_{zyx}} = \rho_{C_z C_{zyx}}$ . Hence,  $\tilde{\sigma}_r(yx) = \tilde{\sigma}_r(y)\tilde{\sigma}_r(x)$ .

Now, we have, for  $x, y \in C_e$  with  $d_{X_{n_r}}(e, yx) < r$ ,  $\tilde{\sigma}_r(y)(\tilde{b}_r(x)) + \tilde{b}_r(y) = \tilde{b}_r(yx)$ . In fact, by noticing that for an affine isometry  $T$  with linear part  $\rho$ ,  $\rho(x - y) = Tx - Ty$ , we have :

$$\begin{aligned}
\rho_{C_z C_{zy}}(c_r^{zy}(x)) &= \rho_{C_z C_{zy}}(t_{C_{zy}}(zy)(s(zy)) - t_{C_{zy}}(zyx)(s(zyx))), \\
&= t_{C_z C_{zy}} \circ t_{C_{zy}}(zy)(s(zy)) - t_{C_z C_{zy}} \circ t_{C_{zy}}(zyx)(s(zyx)) + t_{C_z}(z)(s(z)) - t_{C_z}(zy)(s(zy)), \\
\rho_{C_z C_{zy}}(c_r^{zy}(x)) &= t_{C_z}(zy)(s(zy)) - t_{C_z}(zyx)(s(zyx))
\end{aligned}$$

since  $t_{C_z C_{zy}} \circ t_{C_{zy}}(zy) = t_{C_z}(zy)$  (by Definition 2.3 3. ii)).

Thus,

$$\rho_{C_z C_{zy}}(c_r^{zy}(x)) + c_r^z(y) = t_{C_z}(z)(s(z)) - t_{C_z}(zyx)(s(zyx)) = c_r^z(yx).$$

It follow that :

$$\tilde{\sigma}_r(y)(\tilde{b}_r(x)) + \tilde{b}_r(y) = \frac{1}{(\#X_{n_r})^{\frac{1}{p}}} \bigoplus_{z \in X_{n_r}} (\rho_{C_z C_{zy}}(c_r^{zy}(x)) + c_r^z(y)) = \tilde{b}_r(yx)$$

which proves our claim.

Now, let  $\sigma_r := \tilde{\sigma}_r \circ \pi_{n_r}$  and  $b_r = \tilde{b}_r \circ \pi_{n_r}$  be the lifts of  $\tilde{\sigma}_r$  and  $\tilde{b}_r$  to the  $r$ -ball  $\{g \in \Gamma \mid d_\Gamma(e_\Gamma, g) < r\}$  of  $\Gamma$  and define  $\sigma_r = Id$ ,  $b_r = 0$  outside the  $r$ -ball of  $\Gamma$ . Then  $\sigma_r$  is a  $r$ -locally isometric representation action of  $\Gamma$  on  $\bigoplus_{X_{n_r}} B$ ,  $b_r$  is a  $r$ -locally cocycle with respect to  $\sigma_r$ . Then the map  $\alpha_r$  such that  $\alpha_r(g) \cdot := \sigma_r(g) \cdot + b_r(g)$  is a  $r$ -locally isometric affine action of  $\Gamma$  on  $\bigoplus_{X_{n_r}} B$  and we have, for  $g \in \Gamma$  with  $d_\Gamma(e_\Gamma, g) < r$  :

$$\rho_1(d_\Gamma(e_\Gamma, g)) \leq \|b_r(g)\|_p \leq \rho_2(d_\Gamma(e_\Gamma, g)).$$

From these local isometric affine actions, we build a global isometric affine action of  $\Gamma$  thanks to Lemma 3.3.

Let  $\mathcal{U}$  be a non principal ultrafilter on  $\mathbb{N}^*$ , and let  $B_{\mathcal{U}}$  be the ultraproduct of the family  $\left(\bigoplus_{X_{n_r}} B\right)_{r \in \mathbb{N}^*}$  with respect to  $\mathcal{U}$ . For each  $r \in \mathbb{N}^*$ ,  $\alpha_r$  is a  $r$ -locally isometric affine action of  $\Gamma$  on  $\bigoplus_{X_{n_r}} B$  and since, for any  $g \in \Gamma$ ,  $\|b_r(g)\|_p \leq \rho_2(d_\Gamma(e_\Gamma, g))$  for all  $r \in \mathbb{N}^*$ ,  $(b_r(g))_{r \in \mathbb{N}^*}$  belongs to  $B_{\mathcal{U}}$ . Hence, by Lemma 3.3, there exists an isometric affine action  $\alpha$  of  $\Gamma$  on  $B_{\mathcal{U}}$  such that  $b : g \mapsto (b_r(g))$  is a cocycle with respect to the linear part of this action. Moreover, since for any  $g \in \Gamma$ ,  $\rho_1(d_\Gamma(e_\Gamma, g)) \leq \|b_r(g)\|_p$  for all  $r$  large enough, we have, for all  $g \in \Gamma$  :  $\rho_1(d_\Gamma(e_\Gamma, g)) \leq \|b(g)\|_{B_{\mathcal{U}}}$ , and thus,  $\alpha$  is proper.

As the class of  $L^p$  spaces is closed under  $p$ -normed powers and ultraproduct, it follows that  $\Gamma$  has property  $PL^p$ .  $\square$

*Proof of Theorem 1.1.* It follows from Corollary 2.6 and Proposition 3.4.  $\square$

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